Compact Hyperbolic Manifolds as Internal Worlds [1]

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Abstract

We comment briefly on some of the advantages and disadvantages of compact hyperbolic manifolds as candidate manifolds for large radii compactifications.

1 Introduction and Description of The Problem

The existence of extra dimensions, beyond the known four, seems to be a crucial ingredient in unifying gravity with the other gauge forces. The only promising quantum theory for gravity, so far, is string theory and its mathematical consistency requires the space-time to have ten dimensions. The low energy field theoretical description of string theory can be used in order to explain our four dimensional world using the standard Kaluza–Klein compactification scenario, where the manifold of the 10-dimensional space time, W, is a tensor product of our 4-dimensional space-time, and an internal 6-dimensional manifold. The conventional radius of compactification in string theory is of order M_{pl}^{-1}, resulting in a 10-dimensional gravity scale comparable to the 4-dimensional Planck Mass, $M_{pl} \sim 10^{19}$GeV.

Inspired by the above theory, the last few years have witnessed an increasing interest in adopting actions in more than four dimensions, however performing the compactification on larger radii, $R \gg M_{pl}^{-1}$. The most interesting feature of such proposals [1] is that it suggests a new way, different from grand unified theories and supersymmetry, to solve the hierarchy problem between the electroweak scale and the four dimensional gravity scale.[3] Since the two scales are related by the volume of the d-dimensional internal space

$$M_{pl}^2 = R^d M^{d+2}$$

As can be read from the above relation, the $4+d$ gravity scale can be lowered, e.g. down to TeV, by adjusting the radius R appropriately.

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[3] Other models with extra dimensions using warp product have been proposed [2], but here we limit our discussion to the standard tensor product case.

As a consequence, the solution of the hierarchy problem will be at the cost of introducing new and unexplicabely small mass parameters to the effective four dimensional theory represented by the inverse radius of compactification, depending on how much one lowers M.

For M $\sim$TeV, this radius can be as big as 10^{13}GeV^{-1} for two extra dimensions, 10^{8}GeV^{-1} for three, 10^{5}GeV^{-1} for four, and so on. This problem becomes milder as the number of extra dimensions increases, however it may be desirable to avoid this shortcoming even for a small number of them. The primary problem is not explaining the smallness of these radii, which is obviously a fine-tuning problem, but rather avoiding their undesirable contributions to well studied observables. Upon compactifying down to four dimensions, one may in general get new degrees of freedom added to the spectrum of the Standard Model. The new states can be purely from the gravitational sector, or have Standard Model Kaluza–Klein excitations in addition (depending on whether the SM interactions are written directly in four dimensions, using the induced metric, or written fully in D dimensions). In any case, the new states, having masses of order $1/R$, might lead to detectable modifications of the existing accelerator data and cosmological observations [3]. This lead to imposing judicious bounds on the parameters of these theories. Whether these bounds are implemented or not, the theories with large extra dimensions experience difficulties in realizing complementary scenarios like the standard Cosmological one. [4]

Recently, it was suggested [4] that adopting a compact hyperbolic manifold (CHM) as the internal world may solve the fine-tuning problem in the radii of compactification, mentioned above, utilizing only certain geometrical properties of the internal manifold, namely its exponentially large volume. In addition, and as a consequence of being able to choose the radius of compactification to be of the order of the gravity scale, the undesirable contributions of the Kaluza–Klein degrees of freedom could be avoided in a natural way (we will discuss this issue in seciotn 4).

Although hyperbolic spaces with finite volume are discussed in string and M-theory [6], their existence as a solution of the Einstein's equation should be examined, depending on the theory at hand and the choice of the metric in the 4-dimensional manifold.

2 Compact Hyperbolic Manifolds

A generic hyperbolic manifold, H^d (such as the upper half d-plane), has two main features of relevance to our discussion:

[4] For example, imposing an upper bound on the reheating temperature in order to avoid overproduction of Kaluza–Klein modes of the graviton, and discrete symmetries in order to prevent a fast proton decay makes it difficult to construct a baryogenesis model. Moreover, recovering the standard Friedmann-Robertson-Walker Universe in 4 dimensions starting from higher dimensional Einstein's equations is not straightforward .

- The volume depends exponentially on the curvature of the manifold.
- Constant negative curvature.

The first feature was pointed out in [4] and seems to offer a more statisfactory solution for the hierarchy problem than the "classical" scenarios with large extra dimensions. The second feature, on the other hand, is particularly attractive in our opinion [8], mainly because negatively curved manifolds admit harmonic spinors, a feature which is not yet well appreciated by physicist investigating models with large extra dimensions. This may be due to the fact that getting massless fermions on a compact manifold does not mean achieving a chiral theory in 4 dimensions. In fact, a further coupling to a topologically non-trivial background will be generally needed in order to get rid of the extra spinorial degrees of freedom (for more details see [7]). In the following we will present in more details how can each of those two properties help in solving *some* of the above mentioned problems. Before we do so, it should be mentioned that a hyperbolic manifold, as it is well known, has an infinite volume. However, it is possible to compactify it, and hence to obtain a CHM with finite volume, by moding out by an appropriate discrete subgroup, Γ, of its isometry group.[5] As an example, let us consider H^2 (the upper-half plane). We can chose the following metric on it

$$g^{(d)}_{ij} dy^i dy^j = \frac{1}{R^2} dr^2 + \sinh^2(r/R) d\theta^2$$

One can get a compact H^2 by moding out by the isometry group $SL(2,Z) \subset SL(2,R)$. A sphere of radius R cut out of H^2 will have the volume:

$$V = 2\pi R^2 [\cosh(L/R) + 2L/R]$$

Where L is the diameter (largest distance) of H^2. The relation between the volume and the curvature of the manifold is known as *rigidity*. For $L > R$, one can write

$$V = 2\pi R^2 \mathrm{e}^{L/R}$$

We argue, as in [4], that in the limit $L > R$ the volume for a generic compact H^d with diameter L is:

$$V = a R^d \mathrm{e}^{(d-1)L/R} \tag{1}$$

where a is a numerical factor, and the curvature $|\mathcal{R}| = R^{-2}$. The relation (1) applies in general, with the exception of $d = 3$, where the rigidity property of CHMs breaks down.

[5] In order to avoid singularities, we chose H^d/Γ with Γ acting freely.

3 Some Phenomenological Advantages of CHMs

In the following we will consider theories of the form

$$W = M_4 \times (H^d/\Gamma)$$

(as we mentioned previously, this ansatz should be verified case wise). The metric will be

$$ds^2_W = g^{(4)}_{\mu\nu}(x)dx^\mu dx^\nu + R^2 g^{(d)}_{ij}(y)dy^i dy^j$$

Where $\mu, \nu = 0, ...3$, $i, j = 1, ...d$.[6]

3.1 Approaching the Hierarchy Problem

Let us start our discussion from the gravity sector (as it is the sector which determines the link between M and M_p). Consider some metric fluctuations and look at the linearized Einstein's equations: $\Delta h_{MN} = \Delta_4 h_{MN}(x, y) + \Delta_{H^d} h_{MN}(x, y) = 0$. Upon compactification, the metric fluctuations will fall into various representations of the four dimensional reparameterization group. Namely, $h_{\mu\nu}(x, y)$ is a graviton in M_4 and scalar in H^d; $h_{i\mu}(x, y)$ is a vector in both M_4 and H^d; and $h_{ij}(x, y)$ is a scalar in M_4 and spin-2 field in H^d. The mass spectrum of the various bosonic fields (including vector bosons) will be determind by the eigenvalues, λ_n (to be identified by the mass^2 in the effective four dimensional theory), of the Laplacian on H^d, i.e we need to solve the eigenvalue problem $\Delta_{H^d}\alpha_n^{MN\cdots} = -\lambda_n \alpha_n^{MN\cdot\cdot}$. Δ_{H^d} acts differently on tensors of different ranks, however there are common features for λ's: they are all discrete, ordered ($\lambda_0 \le \lambda_1, ...$), and bounded from below (being eigenvalues of a Laplacian on a compact manifold). The zero mode of the Laplacian on compact space acting on a scalar field, $\Delta_{H^d}\phi(y) = \dfrac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_i\phi(y)\right)$, is constant. Therefore, the wave function of the *massless* graviton in H^d is constant and the effective M_p depends only on the volume factor. Using equation (1) we can write:

$$M_p^2 = M_*^{d+2}V = M_*^{d+2}R^d \exp((d-1)L/R) \tag{2}$$

By mildly tuning the diameter $L \simeq 35M_*^{-1}$ (10^{-15} mm), the above equation (2) represents a good solution for the hierarchy problem, at least at the classical level.

3.2 Harmonic Spinors

If one wants to end up with massless fermions in 4 dimensions after compactification, so that the standard model fermions get their masses through a Higgs mechanism, it is necessary for the Dirac operator on the internal manifold, $\not{D}$, to

[6] For other phenomenological and cosmological implications see [4, 5].

have at least one zero mode.[7] As we pointed out in [8], using earlier theorem by Lichnorowicz [9], positively curved compact manifolds do *not* admit harmonic spinors, while negatively curved compact do. This can be easily understood looking at the eigenvalues of $\not{D}^2$ (since $\ker\not{D}^2 = \ker\not{D}$)

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathcal{R}$$

Where $\mathcal{R}$ is the scalar curvature, and $\nabla^*\nabla$ is the connection Laplacian (a positive operator).

As can be easily verified, a manifold with a positive curvature, like S^2 for example, does not admit harmonic (massless) spinors. Inorder to be able to get massless spinors on a sphere, it is necessary to couple the spinors to a magnetic monopole.[8] In general, one has to do an extra labor in order to get massless spinors by compactifying on positively curved manifolds, e.g. by twisting the Dirac operator of the internal space, or moding by its isometry group. On the other hand, one can generate naturally massless fermions by compactifying on manifolds of negative curvature, like the CHMs.

4 Some Phenomenological Disadvantages of CHMs

There is no analytical expression for the eigenvalues of the Laplacian on a generic compact manifold. Reliance on mere dimensional analysis to set lower bounds may break down in some cases. In the work [8] we used earlier results by [10] and pointed out lower bounds on the first eigenvalue of the Laplacian acting on a scalar on generic compact manifold Y:

$$\lambda_1 \geq \frac{\pi^2}{4L^2} - \max\{-(d-1)K, 0\} \tag{3}$$

where $\min\mathcal{R} = (d-1)K$ (for constant curvature, $\mathcal{R} = (d-1)K$). For $\mathcal{R} > 0$, the fundamental parameter of the theory is the diameter, L (the maximum distance on the manifold). For $\mathcal{R} < 0$ the curvature will also enter into the bounds. In CHMs, within the approximation $L > R$ used to solve the hierarchy problem, there seem to be no lower bound on the first massive Kaluza–Klein graviton mode from geometry, namely because the mass^2 will be bounded from below by a negative number $m^2_{KK} \geq -\frac{1}{R^2}$.

Of course, this does not mean that the values of m^2_{KK} are not of the order TeV. The fact that there exists a mass gap, makes this assumption reasonable.

We went further to discuss lower bounds on fermionic Kaluza-Klein modes for a generic compact Y. The first non-zero eigenvalue of $\not{D}_Y$ on a compact

[7] Assuming that the compactified manifold admits a spin structure.

[8] I am grateful to Seif Randjbar-Daemi for very useful discussions around this point.

space is bounded from below by [11]

$$m_e^2 \geq \frac{d}{4(d-1)}\tau_1 \tag{4}$$

Where τ_1 is the first non-zero eigenvalue of the Yamabe operator $\mathcal{L} = \dfrac{4(d-1)}{d-2}\Delta_Y + \mathcal{R}$. Using the bound (3) it is possible to see that for CHMs there are no geometric lower bounds on the fermions masses, within the approximation $L > R$.

5 Conclusions

A compact hyperbolic manifold, $H^d; d \neq 3$, if proven to exist as a solution of the equations of motion for a particular geometry of the 4 dimensional Universe, has some attractive features which can be used in favor of the theories with large radii, compactification: i) if the diameter of the CHM is slightly larger than the scale of its curvature, it is possible to have a rather satisfactory *classical* solution for the hierarchy problem since only a very mild tuning for the parameters of the manifold is required ii) CHMs admit harmonic spinors iii) all the Kaluza–Klein excitations of the metric are heavy [4] $\geq 1/L$ (this issue was not discussed in this talk). Unfortunately, as can be deduced from (3) and (4), there are no geometric lower bounds on the masses of Kaluza–Klein modes of the gravity, fermions, and scalar fields. Hence relaxing the known bounds (like the upper bound on the reheating temperature and other astrophysical bounds) is not geometrically justified unless a further case-wise investigation is performed (see [4] for the case of H^2).

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